Generalization of Some Improved Integral Methods for Transient Heat Conduction Problems

H. S. Kou* and N. W. Fan†
Tatung Institute of Technology, Taiwan, Republic of China

Abstract

THE purpose of this investigation is to derive the conventional heat-balance integral method and some other improved integral methods into the general formulation for solving the approximate solutions of transient heat conduction problems. The generalized x moment, θ moment, and Zien[1] treatments are introduced, respectively, in the present study. Then, the assumed thermal profile with the exponential or binomial form for the transient heat conduction problem with prescribed boundary temperature or prescribed boundary heat flux is induced into one of the preceding generalized integral methods. Results from the various methods for each thermal profile and boundary condition are presented to observe the effects of the exponent of multiplier. Besides, the more suitable choice of the exponent for the range of accuracy is suggested to minimize the error due to the approximate technique and enhance its reliability and effectiveness.

Contents

Consider the standard governing equation of the one-dimensional transient heat conduction problem in a semi-infinite medium:

$$\frac{\partial\theta}{\partial t} = \alpha\frac{\partial^2\theta}{\partial x^2}, \qquad t \geq 0,\ 0 \leq x < \infty \tag{1}$$

where the thermal properties of the solid are assumed constant to simplify the problem.

Generalized θ-Moment Treatment

In the present treatment, Eq. (1) must be cast in generalized integral form. This form can be obtained by multiplying Eq. (1) by θ^k, where k is an arbitrary constant, and integrating with respect to the space variable over the whole domain. With the usual boundary conditions of $\theta_\infty = (\partial\theta/\partial x)_\infty = 0$, it is reduced to

$$\int_0^\infty \frac{\partial\theta^{k+1}}{\partial t}\,\mathrm{d}x = -(k+1)\alpha\theta_0^k\left(\frac{\partial\theta}{\partial x}\right)_0 - (k+1)k\alpha\int_0^\infty \theta^{k-1}\left(\frac{\partial\theta}{\partial x}\right)^2\mathrm{d}x \tag{2}$$

Generalized Zien's Treatment

The following is a more general form of Zien's approach. Two equations are obtained by multiplying Eq. (1) with θ^k and θ^ℓ, respectively, and then integrating over the entire region of interest. Note here that ℓ and k are arbitrary constants. With the usual boundary conditions of $\theta_\infty = (\partial\theta/\partial x)_\infty = 0$, the two generalized integral equations are written as

$$\int_0^\infty \frac{\partial\theta^{k+1}}{\partial t}\,\mathrm{d}x = -(k+1)\alpha\theta_0^k\left(\frac{\partial\theta}{\partial x}\right)_0 - (k+1)k\alpha\int_0^\infty \theta^{k-1}\left(\frac{\partial\theta}{\partial x}\right)^2\mathrm{d}x \tag{3}$$

and

$$\int_0^\infty \frac{\partial\theta^{\ell+1}}{\partial t}\,\mathrm{d}x = -(\ell+1)\alpha\theta_0^\ell\left(\frac{\partial\theta}{\partial x}\right)_0 - (\ell+1)\ell\alpha\int_0^\infty \theta^{\ell-1}\left(\frac{\partial\theta}{\partial x}\right)^2\mathrm{d}x \tag{4}$$

In this general form, Zien's method becomes the special case where $k = 0$ for the first integral equation and $\ell = 1$ for the second.

In the case of boundary condition with prescribed boundary temperature, combining Eq. (3) with Eq. (4) by eliminating boundary heat flux, the modified thermal penetration depth which comes from the assumed temperature profile can be solved. In order to get higher accuracy, the boundary heat flux is not to be obtained directly by determining the slope of the temperature profile on the surface. Instead, it must be obtained by substituting the calculated temperature profile or modified thermal penetration depth into either one of the preceding equations.

In another case with prescribed boundary heat flux, a profile that contains two parameters and satisfies only the boundary condition of $\theta_\infty = 0$ is adopted. Note here that the boundary heat flux $(\partial\theta/\partial x)_o$ in Eqs. (3) and (4) is not obtained from the derivative of the assumed temperature profile. Instead, it is substituted directly from the exactly prescribed boundary heat flux to reduce the error. After introducing the assumed temperature profile into Eqs. (3) and (4), respectively, two simultaneous ordinary differential equations that are used to determine the two parameters are then derived.

Generalized x-Moment Treatment

In this scheme, a generalized form is obtained by multiplying Eq. (1) by x^k and then integrating the space variable over the whole spatial interval. With the usual boundary conditions of $\theta_\infty = (\partial\theta/\partial x)_\infty = 0$, it reduces to

$$\int_0^\infty x^k\frac{\partial\theta}{\partial t}\,\mathrm{d}x = \alpha k(k-1)\int_0^\infty x^{k-2}\theta\,\mathrm{d}x, \qquad \text{for } k \neq 0 \tag{5}$$

and

$$\int_0^\infty \frac{\partial\theta}{\partial t}\,\mathrm{d}x = -\alpha\left(\frac{\partial\theta}{\partial x}\right)_0, \qquad \text{for } k = 0 \tag{6}$$

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*Associate Professor, Department of Mechanical Engineering.
†Graduate Student, Department of Mechanical Engineering.

Table 1 Comparison between exact and integral solutions

n		$Q = \left(\frac{\partial\theta}{\partial x}\right)_0 \frac{\sqrt{\alpha t}}{\theta_t}$ 0	1	2	3	$\Phi = \frac{\kappa\theta_0}{q_0\sqrt{\alpha t}}$ 0	1	2	3
Exact		0.5642	0.8862	1.1284	1.3293	1.1284	0.8862	0.7523	0.6647
HBI		0.7072	1	1.2247	1.4142	1	0.8165	0.7071	0.6325
Generalized θ-moment treatment ($k = 0.55$)	Exponential	0.5680	0.9070	1.1500	1.3500	1.1026	0.8695	0.7407	0.6562
	Binomial, $m = 3$	0.5591	0.8928	1.1321	1.3290	1.1201	0.8833	0.7525	0.6666
	Binomial, $m = 16$	0.5634	0.8997	1.1408	1.3392	1.1115	0.8766	0.7467	0.6615
Generalized x-moment treatment ($k = 0.55$)	Exponential	0.5679	0.8856	1.1259	1.3260	1.1292	0.8882	0.7542	0.6663
	Binomial, $m = 3$	0.5707	0.8898	1.1312	1.3323	1.1238	0.8840	0.7506	0.6632
	Binomial, $m = 16$	0.5693	0.8878	1.1287	1.3293	1.1263	0.8860	0.7523	0.6647
Generalized Zien's treatment ($k = 0.9$, $\ell = 0$)	Exponential	0.5723	0.8898	1.1297	1.3295	1.1239	0.8852	0.7522	0.6649
	Binomial, $m = 3$	0.5605	0.8896	1.1400	1.3488	1.1240	0.8772	0.7414	0.6532
	Binomial, $m = 16$	0.5677	0.8863	1.1273	1.3281	1.1283	0.8870	0.7530	0.6652

In the case of prescribed temperature boundary condition, the profile containing one unknown parameter is introduced into Eq. (5) to solve the parameter. The boundary heat flux term is eliminated in Eq. (5) by inserting the boundary conditions and then reducing to $x^k(\partial\theta/\partial x)|_0^\delta = x^{k-1}\theta|_0^\delta = 0$. Therefore, there is no substitution into Eq. (6) from Eq. (5). The boundary heat flux needs to be calculated by using Eq. (6), which includes the boundary heat flux term with integral representation.

In the case of prescribed heat flux boundary condition, the treatment is similar to that of Zien's treatment. The remarkable thing is that this treatment reduces to the conventional heat-balance integral method as k equals zero. It also gives the same results with the double-integral methods developed by Volkov[2] where k is equal to one.

Applications

It is known that clever selection to find the suitable values of the exponent for the preceding three different treatments can reduce the discrepancy between exact and approximate solutions. In order to understand the influence of exponents more clearly, the following cases have been used to validate the effectiveness and also find the optimal ranges.

Prescribed Power-Law Surface Temperature to Find Surface Heat Flux

Consider the boundary condition with $\theta_0 = at^{n/2}$, $(n \geq 0)$. The approximate thermal profiles are assumed to be

$$F = at^{n/2}e^{-x/\delta} \tag{7}$$

for the exponential profile and

$$F = at^{n/2}(1 - x/\delta)^m \tag{8}$$

for the binomial profile, which satisfy the essential boundary conditions of $\theta_0 = at^{n/2}$ and $\theta_\infty = 0$.

Prescribed Power-Law Surface Heat Flux to Find Surface Temperature

Consider the boundary condition with $q_o = -\kappa(\partial\theta/\partial x)_0 = at^{n/2}$, $(n \geq 0)$. The approximate temperature profiles involving only one parameter are assumed to be

$$F = \frac{q_0\delta}{\kappa} e^{-x/\delta} \tag{9}$$

for the exponential profile and

$$F = \frac{q_0\delta}{\kappa}\left(1 - \frac{x}{\delta}\right)^m \tag{10}$$

for the binomial profile for the generalized θ-moment scheme, which satisfy the essential boundary conditions of $-\kappa(\partial\theta/\partial x)_0 = at^{n/2}$ and $\theta_\infty = 0$.

For the generalized Zien method as well as the generalized x-moment scheme, the profiles containing two parameters are assumed as

$$F = \frac{q_0\delta}{\kappa}\beta e^{-x/\delta} \tag{11}$$

for the exponential profile and

$$F = \frac{q_0\delta}{\kappa}\beta\left(1 - \frac{x}{\delta}\right)^m \tag{12}$$

for the binomial profile, which satisfy only the boundary condition of $\theta_\infty = 0$.

Conclusions

The general form of the integral method for the θ-moment scheme, Zien's treatment, and the x-moment scheme has been introduced. Four different cases have been used to see the effects of the exponents and also the assumed thermal profiles. Table 1 shows the results for the comparison of the exact solution[3] with the approximate solutions when optimal values of k and ℓ are chosen.

For the generalized θ-moment scheme, the suitable values of the exponent are $1 \geq k \geq 0.4$, where the higher accuracy can be achieved. It is also found that in the classical HBI method, $k = 0$, which is far beyond the range of optimal values and gives worse precision.

The generalized x-moment scheme tends to Volkov's method as $k = 1$. The appropriate value of the exponent is $0.5 \leq k \leq 0.6$. Therefore, Volkov's method does not derive the optimal results.

The generalized Zien's treatment usually gives better accuracy than the generalized θ- or x-moment schemes. We have shown that suitable values of the exponents are $1 \geq k \geq 0.8$ and $1 \geq \ell \geq 0$. Therefore, Zien's method can get quite good results.

References

[1]Zien, T. F., "Approximate Calculation of Transient Heat Conduction," *AIAA Journal*, Vol. 14, No. 3, 1976, pp. 404-406.

[2]Volkov, V. N., "A Refinement of the Karman-Pohlhausen Integral Method in Boundary-Layer Theory," *Journal of Engineering Physics*, Vol. 9, No. 5, 1965, pp. 371-374.

[3]Carslaw, H. S., and Jaeger, J. C., *Conduction of Heat in Solids*, 2nd ed., Oxford Univ., London, UK, 1959, Chap. 2.